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# Colorings of simplicial complexes and vector bundles over Davis–Januszkiewicz spaces

Dietrich Notbohm

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**Abstract** We show that coloring properties of a simplicial complex  $K$  are reflected by splitting properties of a bundle over the associated Davis–Januszkiewicz space whose Chern classes are given by the elementary symmetric polynomials in the generators of the Stanley–Reisner algebra of  $K$ .

**Keywords** Davis–Januszkiewicz spaces · Vector bundle · Characteristic classes · Colorings · Simplicial complexes

**Mathematics Subject Classification (2000)** 55R10 · 57R22 · 05C15

## 1 Introduction

For a simplicial complex  $K$ , Davis and Januszkiewicz constructed a family of spaces, all of which are homotopy equivalent, and whose integral cohomology is isomorphic to the associated Stanley–Reisner algebra  $\mathbb{Z}[K]$  [2, Sect. 4]. We denote a generic model for this homotopy type by  $DJ(K)$ . In the above mentioned influential paper, Davis and Januszkiewicz also constructed a particular complex vector bundle  $\lambda$  over  $DJ(K)$  whose Chern classes are given by the elementary symmetric polynomials in the generators of  $\mathbb{Z}[K]$  [2, Sect. 6]. This vector bundle is of particular interest. For example, if  $K$  is the dual of the boundary of a simple polytope  $P$ , then the associated moment angle complex  $Z_K$  is a manifold and the realification  $\lambda_{\mathbb{R}}$  of  $\lambda$  is stably isomorphic to the bundle given by applying the Borel construction to the tangent bundle of  $Z_K$ . And if  $M^{2n}$  is a quasitoric manifold over  $P$ , then again the Borel construction applied to the tangent bundle of  $M^{2n}$  produces a vector bundle stably isomorphic to  $\lambda_{\mathbb{R}}$  [2, Theorem 6.6, Lemma 6.5].

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Davis and Januszkiewicz also noticed that, if  $K$  is the dual of the boundary of a polytope of dimension  $n$  and admits a coloring with  $n$  colors, the bundle  $\lambda$  splits into a direct sum of  $n$  complex line bundles and a trivial bundle [2, Sect. 6.2]. We are interested in generalizations of this observation. In fact, we will show that a simplicial complex admits a coloring with  $r$  colors precisely when  $\lambda$  splits stably into a direct sum of  $r$  linear complex bundles and a trivial bundle. We will also show that a similar result holds for the realification of  $\lambda$ .

To make our statements more precise we have to fix notation and recall some basic constructions. Let  $[m] := \{1, \dots, m\}$  be the set of the first  $m$  natural numbers. A finite abstract simplicial complex  $K$  on  $[m]$  is given by a set of faces  $\alpha \subseteq [m]$  which is closed under the formation of subsets. We consider the empty set  $\emptyset$  as a face of  $K$ . The dimension  $\dim \alpha$  of a face  $\alpha$  is given in terms of its cardinality by  $|\alpha| - 1$ , and the dimension  $\dim K$  of  $K$  is the maximum of the dimensions of its faces. The most basic examples are given by full simplices. For  $\alpha \subseteq [m]$  we denote by  $\Delta[\alpha]$  the simplicial complex which consists of all possible subsets of  $\alpha$ . Then  $\Delta[\alpha]$  is an  $(|\alpha| - 1)$ -dimensional simplex. The full simplex  $\Delta[m]$  contains  $K$  as a subcomplex, and if  $\alpha \in K$  then  $\Delta[\alpha] \subset K$  is a subcomplex as well.

A *regular  $r$ -paint coloring*, an  *$r$ -coloring* for short, of a simplicial complex  $K$  is a non degenerate simplicial map  $g: K \rightarrow \Delta[r]$ , i.e.  $g$  maps each face of  $K$  isomorphically on a face of  $\Delta[r]$ . The inclusion  $K \subset \Delta[m]$  always provides an  $m$ -coloring. If  $\dim(K) = n - 1$ , then  $K$  may only allow  $r$ -colorings for  $r \geq n$ .

For a commutative ring  $R$  with unit we denote by  $R[m] := R[v_1, \dots, v_m]$  the graded polynomial algebra generated by the algebraically independent elements  $v_1, \dots, v_m$  of degree two, one for each vertex of  $K$ . For each subset  $\alpha \subseteq [m]$  we denote by  $v_\alpha := \prod_{j \in \alpha} v_j$  the square-free monomial whose factors are in one-to-one correspondence with vertices contained in  $\alpha$ . The graded Stanley-Reisner algebra  $R[K]$  associated with  $K$  is defined as the quotient  $R[K] := R[m]/I_K$ , where  $I_K \subset R[m]$  is the ideal generated by all elements  $v_\mu$  such that  $\mu \subseteq [m]$  is not a face of  $K$ .

Since  $BT^m$  is an Eilenberg-MacLane space realizing the polynomial algebra  $\mathbb{Z}[m]$ , the projection  $\mathbb{Z}[m] \rightarrow \mathbb{Z}[K]$  can be realized by a map  $f: DJ(K) \rightarrow BT^m$ . We can think of  $T^m$  as the maximal torus of the unitary group  $U(m)$ . The pull back along the composition  $DJ(K) \rightarrow BT^m \rightarrow BU(m)$  of the universal bundle over  $BU(m)$  gives a vector bundle  $\lambda \downarrow DJ(K)$ . This is the vector bundle studied by Davis and Januszkiewicz and mentioned above. The total Chern class  $c(\lambda) = 1 + \sum_i c_i(\lambda)$  of  $\lambda$  is then given by  $c(\lambda) = \prod_{i=1}^m (1 + v_i) \in \mathbb{Z}[K]$ .

The realification of a complex vector bundle  $\xi$  is denoted by  $\xi_{\mathbb{R}}$ . Confusing notation we will denote by  $\mathbb{C}$  and  $\mathbb{R}$  a 1-dimensional trivial complex or real vector bundle over a space  $X$ . Now we can state our main theorem.

**Theorem 1.1** *Let  $K$  be an finite simplicial complex over the vertex set  $[m]$ . Then the following conditions are equivalent.*

- (i)  $K$  admits an  $r$ -coloring  $K \rightarrow \Delta[r]$ .
- (ii) The vector bundle  $\lambda$  splits into a direct sum  $(\bigoplus_{i=1}^r v_i) \oplus \mathbb{C}^{m-r}$  of  $r$  complex line bundles  $v_i$  and a trivial  $(m - r)$ -dimensional complex bundle.
- (iii) The realification  $\lambda_{\mathbb{R}}$  of  $\lambda$  splits into a sum  $(\bigoplus_{i=1}^r \theta_i) \oplus \mathbb{R}^{2(m-r)}$  of  $r$  2-dimensional real bundles  $\theta_i$  and a trivial  $2(m - r)$ -dimensional real bundle.
- (iv) The vector bundle  $\lambda$  is stably isomorphic to a direct sum  $\bigoplus_{i=1}^r v_i$  of  $r$  complex line bundles.
- (v) The realification  $\lambda_{\mathbb{R}}$  is stably isomorphic to a direct sum  $\bigoplus_{i=1}^r \theta_i$  of  $r$  2-dimensional real bundles.

Several of our vector bundles will be constructed as homotopy orbit spaces. For a compact Lie group  $G$  and a  $G$ -space  $X$ , the *Borel construction* or *homotopy orbit space*  $EG \times_G X$  will be denoted by  $X_{hG}$ . If  $\eta \downarrow X$  is an  $n$ -dimensional  $G$ -vector bundle over  $X$  with total space  $E(\eta)$ , the Borel construction establishes a fibre bundle  $E(\eta)_{hG} \rightarrow X_{hG}$ . In fact, this is an  $n$ -dimensional vector bundle over  $X_{hG}$ , denoted by  $\eta_{hG}$ . For definitions and details see [4].

Let  $M^{2n}$  be a quasitoric manifold over the simple polytope  $P$ . That is that  $M^{2n}$  carries a  $T^n$ -action, which is locally standard and that the orbit space  $M^{2n}/T^n = P$  is a simple polytope. The Borel construction produces a space  $(M^{2n})_{hT^n}$ , which is homotopy equivalent to  $DJ(K_P)$ , where  $K_P$  is the simplicial complex dual to the boundary of  $P$ . For details see [2, Sect. 4.2]. Let  $\tau_M$  denote the tangent bundle of  $M^{2n}$ . Davis and Januszkiewicz showed that the vector bundle  $(\tau_M)_{hT^n} \downarrow DJ(K)$  and  $\lambda_{\mathbb{R}}$  are stably isomorphic as real vector bundles over  $DJ(K_P)$  [2, Sect. 6]. We can draw the following corollary of Theorem 1.1.

**Corollary 1.2** *Let  $M^{2n}$  be a quasitoric manifold over a simple polytope  $P$ . Let  $K_P$  be the simplicial complex dual to the boundary of  $P$ . If the tangent bundle  $\tau_M$  of  $M^{2n}$  is stably equivariantly isomorphic to a direct sum of  $r$  2-dimensional real equivariant  $T^n$ -bundles over  $M^{2n}$ , then  $K_P$  admits an  $r$ -coloring.*

The paper is organized as follows. For the proof of our main theorem we will need two different models for  $DJ(K)$ . They are discussed in the next section. In Sect. 3, we will use some geometric constructions to produce a splitting of  $\lambda$  from a given coloring. The final section contains the proof of Theorem 1.1.

If not specified otherwise,  $K$  will always denote an  $(n - 1)$ -dimensional finite simplicial complex with  $m$ -vertices.

## 2 Models for $DJ(K)$

Let  $\text{CAT}(K)$  denote the category whose objects are the faces of  $K$  and whose arrows are given by the subset relations between the faces.  $\text{CAT}(K)$  has an initial object given by the empty face. Given a pair  $(X, Y)$  of pointed topological space we can define covariant functors

$$X^K, (X, Y)^K : \text{CAT}(K) \rightarrow \text{TOP}.$$

The functor  $X^K$  assigns to each face  $\alpha$  the cartesian product  $X^\alpha$  and to each morphism  $i_{\alpha, \beta}$  the inclusion  $X^\alpha \subset X^\beta$  where missing coordinates are set to the base point  $*$ . If  $\alpha = \emptyset$ , then  $X^\alpha$  is a point. And  $(X, Y)^K$  assigns to  $\alpha$  the product  $X^\alpha \times Y^{[m] \setminus \alpha}$  and to  $i_{\alpha, \beta}$  the coordinate wise inclusion  $X^\alpha \times Y^{[m] \setminus \alpha} \subset X^\beta \times Y^{[m] \setminus \beta}$ . The inclusions  $X^\alpha \subset X^{[m]} = X^m$  and  $X^\alpha \times Y^{[m] \setminus \alpha} \subset X^m$  establish inclusions

$$\text{colim}_{\text{CAT}(K)} X^K \rightarrow X^m, \quad \text{colim}_{\text{CAT}(K)} (X, Y)^K \rightarrow X^m.$$

We are interested in two particular cases, namely the functor  $X^K$  for the classifying space  $BT = \mathbb{C}P^\infty$  of the 1-dimensional circle  $T$  and the functor  $(X, Y)^K$  for the pair  $(D^2, S^1)$ . The colimit

$$Z_K := \text{colim}_{\text{CAT}(K)} (D^2, S^1)^K$$

is called the *moment angle complex* associated to  $K$ . The inclusions  $Z_K \subset (D^2)^m \subset \mathbb{C}^m$  allow to restrict the standard  $T^m$ -action on  $\mathbb{C}^m$  to  $Z_K$ . The Borel construction produces a fibration

$$q_K : (Z_K)_{hT^m} \rightarrow BT^m$$

with fiber  $Z_K$ . Moreover,  $B_T K := (Z_K)_{hT^m}$  is a realization of the Stanley-Reisner algebra  $\mathbb{Z}[K]$  and a model for  $DJ(K)$ . That is there exists an isomorphism  $H^*(B_T K; \mathbb{Z}) \cong \mathbb{Z}[K]$  such that the map  $H^*(q_K; \mathbb{Z})$  can be identified with the map  $\mathbb{Z}[m] \rightarrow \mathbb{Z}[K]$  [2, Theorem 4.8]. We will use this model for geometric construction with our vector bundles.

Buchstaber and Panov gave a different construction for  $DJ(K)$ . They showed that  $c(K) := \text{colim}_{\text{CAT}(K)} BT^K$  is homotopy equivalent to  $B_T K$  and that the inclusion

$$c(K) \rightarrow BT^m$$

is homotopic to  $q_K$  [1, Theorem 6.29]. In particular, each face  $\alpha \in K$  defines a map  $h_\alpha: BT^\alpha \rightarrow c(K)$ . The model  $c(K)$  will be used to produce a coloring from a stable splitting of  $\lambda$ .

**Remark 2.1** If  $K$  is the triangulation of an  $(n - 1)$ -dimensional sphere, the moment angle complex  $Z_K$  is a manifold. In this case, the tangent bundle  $\tau_Z$  is a  $(m + n)$ -dimensional  $T^m$ -equivariant vector bundle, which satisfies the analogue of Corollary 1.2. If  $\tau_Z$  is stably equivariantly isomorphic to a direct sum of  $r$  2-dimensional real equivariant  $T^m$ -bundles over  $Z_K$ , then  $K$  admits an  $r$  coloring. Again this follows from the fact that  $(\tau_Z)_{hT^m}$  and  $\lambda_{\mathbb{R}}$  are stably isomorphic [2, Sect. 6].

### 3 Geometric constructions

The  $m$ -dimensional torus  $T^m$  acts coordinate wise on  $\mathbb{C}^m$ . And the diagonal action of  $T^m$  on  $\mathbb{C}^m \times Z_K$  makes the projection  $\mathbb{C}^m \times Z_K \rightarrow Z_K$  onto the second factor into a  $T^m$ -equivariant complex vector bundle over  $Z_K$ , denoted by  $\lambda'$ . An application of the Borel construction produces the bundle  $\lambda := \lambda'_{hT^m} \downarrow B_T K$  over  $B_T K$  whose total Chern class is given by  $c(\lambda) = \prod_i (1 + v_i) \in \mathbb{Z}[K]$  and whose classifying map is the composition  $B_T K \xrightarrow{q_K} BT^m \rightarrow BU(m)$ . Since  $T^m$  acts coordinatewise on  $\mathbb{C}^m$ , both bundles,  $\lambda'$  and  $\lambda$  split into a direct sum of (equivariant) line bundles. Let  $\mathbb{C}_j$  denote the  $j$ th component of  $\mathbb{C}^m$ . In particular,  $T^m$  acts on  $\mathbb{C}_j$  via the projection  $T^m \rightarrow T^{\{j\}}$  onto the  $j$ th component of  $T^m$ . The vector bundle  $\lambda'_j := \mathbb{C}_j \times Z_K$  is  $T^m$ -equivariant, and  $\lambda_j := (\lambda'_j)_{hT^m}$  is a 1-dimensional complex vector bundle over  $B_T K$ . We have  $\lambda' \cong \bigoplus_j \lambda'_j$  and  $\lambda \cong \bigoplus_j \lambda_j$ . All this can be found in [2, Sect. 6].

If  $g: K \rightarrow \Delta[r]$  is an  $r$ -coloring we want to construct an equivariant splitting of  $\lambda' \downarrow Z_K$  into a direct sum of  $T^m$ -equivariant complex line bundles and a trivial bundle  $\mathbb{C}^{m-r}$ . We will use ideas of Davis and Januszkiewicz discussed in [2, Sect. 6.2]. For each  $i \in [r]$  we denote by  $S_i := g^{-1}(i) \subset [m]$  the preimage of  $i$  and by  $s_i := |S_i|$  the order of  $S_i$ . There are two vector bundles associated with  $S_i$ , namely the tensor product  $v_i := \bigotimes_{j \in S_i} \lambda'_j$  of all complex line bundles associated to the vertices contained in  $S_i$  and the direct sum  $\eta_i := \bigoplus_{j \in S_i} \lambda'_j$  of all these line bundles. Both are  $T^m$ -equivariant vector bundles over  $Z_K$ .

**Lemma 3.1** *For all  $i \in [r]$ , there exists an  $T^m$ -equivariant vector bundle isomorphism  $v_i \oplus \mathbb{C}^{s_i-1} \rightarrow \eta_i$ .*

For simplicial complexes dual to the boundary of simple polytopes the claim is already stated in [2, Sect. 6.2]. We will give here a different proof.

*Proof* For simplification we drop the subindex  $i$  in the notation and assume that  $S = [s]$ . We will think of  $\mathbb{C}^{s-1} \subset \mathbb{C}^s$  as the subspace given by  $\{(x_1, \dots, x_s) \in \mathbb{C}^s \mid \sum_k x_k = 0\}$ . We define a map

$$f : \mathbb{C} \times \mathbb{C}^{s-1} \times Z_K \longrightarrow \mathbb{C}^s \times Z_K$$

by  $f(y, x, z) := (u, z)$  where the  $j$ th coordinate  $u_j$  of  $u$  is given by  $u_j := y \prod_{k \neq j, k \in [s]} \bar{z}_k + z_j x_j$ . Here,  $\bar{z}_k$  denotes the complex conjugate of  $z_k$ . If  $T^m$  acts on  $\mathbb{C}$  via the map  $t \mapsto \prod_{j \in [s]} t_j$ , trivially on  $\mathbb{C}^{s-1}$  and on  $\mathbb{C}^s$  via the projection  $T^m \longrightarrow T^s$  onto the first  $s$  coordinates, one can easily show that this map is  $T^m$ -equivariant. Moreover, with these actions, the source is the total space of the bundle  $\nu \oplus \mathbb{C}^{s-1} \downarrow Z_K$  and the target the total space of  $\eta \downarrow Z_K$ . Since both sides have the same dimension, it is only left to show that  $f$  is fiber wise a monomorphism.

By construction, any subset  $\{j, k\} \subset [s]$  is a missing face in  $K$ . Since  $Z_K = \bigcup_{\alpha \in K} (D^2)^\alpha \times (S^1)^{[m] \setminus \alpha}$ , the space  $(D^2)^{\{j, k\}} \times (S^1)^{[m] \setminus \{i, j\}}$  is not contained in  $Z_K$  and for  $z = (z_1, \dots, z_m) \in Z_K$  there is at most one coordinate among  $z_1, \dots, z_s$  which is trivial.

Now we assume that  $f(y, x, z) = (0, z)$ . In particular, we have  $x_j z_j = -y \prod_{k \neq j} \bar{z}_k$ . If one of the coordinates  $z_j$  vanishes, say  $z_1 = 0$ , then  $z_j \neq 0$  for  $j \neq 1$  and hence  $y = 0$  as well as  $x_j = 0$  for  $j \neq 1$ . Since  $\sum_j x_j = 0$ , we also have  $x_1 = 0$ .

If  $z_j \neq 0$  for all  $j$ , then  $x_j = y \prod_{k \neq j} \bar{z}_k / z_j$  and  $0 = \sum_j x_j = \sum_j y \prod_{k \neq j} \bar{z}_k / z_j = y \sum_j \prod_{k \neq j} \bar{z}_k / z_j$ . Multiplying with  $\prod_j z_j$  shows that  $y \sum_j \prod_{k \neq j} \bar{z}_k z_k = 0$  and hence that  $y = 0$  as well as  $x_j = 0$  for all  $j$ . This shows that  $f$  is a fiber wise monomorphism and finishes the proof.  $\square$

**Corollary 3.2** *Let  $K \longrightarrow \Delta[r]$  be an  $r$ -coloring of a finite simplicial complex. Then the following holds:*

- (i) *The bundle  $\lambda' \downarrow Z_K$  splits equivariantly into a direct sum of  $r$  equivariant complex line bundles and a trivial bundle.*
- (ii) *The bundle  $\lambda \downarrow DJ(K)$  splits into a direct sum of  $r$  complex line bundles and a trivial bundle.*

*Proof* By Proposition 3.1 we have

$$\lambda' \cong \bigoplus_{j=1}^r \bigoplus_{i \in S_j} \lambda'_i \cong \bigoplus_{j=1}^r \left( \nu'_j \oplus \mathbb{C}^{s_i-1} \right) \cong \left( \bigoplus_{j=1}^r \nu'_j \right) \oplus \mathbb{C}^{m-r}.$$

This proves the first part, the second follows from the first by applying the Borel construction.  $\square$

#### 4 Proof of Theorem 1.1

The proof needs some preparation. For topological spaces  $X$  and  $Y$  we denote by  $[X, Y]$  the set of homotopy classes of maps from  $X$  to  $Y$  and for two compact Lie groups  $G$  and  $H$  by  $\text{hom}(H, G)$  the set of Lie group homomorphism  $H \longrightarrow G$ .

Let  $G$  be a compact connected Lie group with maximal torus  $j : T_G \hookrightarrow G$  and Weyl group  $W_G$ . Since the action of  $W_G$  on  $T_G$  is induced by conjugation with elements of  $G$ , the composition of  $w \in W_G$  and  $j$  induces a map between the classifying spaces homotopic to  $Bj$ . And passing to classifying spaces followed by composing with  $Bj$  induces a map  $\text{hom}(H, T_G) \longrightarrow [BH, BG]$  which factors through the orbit space of the  $W_G$ -action on  $\text{hom}(H, T_G)$  and provides a map  $\text{hom}(H, T_G)/W_G \longrightarrow [BH, BG]$ . The following two facts may be found in [3] and are needed for the proof of our main theorem.

**Theorem 4.1** [3] *Let  $G$  be a connected compact Lie group and  $S$  a torus.*

- (i) *The map  $\text{hom}(S, T_G)/W_G \rightarrow [BS, BG]$  is a bijection.*
- (ii) *The map  $[BS, BG] \rightarrow \text{Hom}(H^*(BG; \mathbb{Q}), H^*(BS; \mathbb{Q}))$  is an injection.*

The rational cohomology  $H^*(BG; \mathbb{Q}) \cong H^*(BT_G, \mathbb{Q})^{W_G}$  is the ring of polynomial invariants of the induced  $W_G$ -action on the polynomial algebra  $H^*(BT_G; \mathbb{Q})$ . For  $G = SO(2k+1)$  the maximal torus  $T_{SO(2k+1)} = T^k$  is an  $k$ -dimensional torus and we can identify  $H^*(BT_{SO(2k+1)}; \mathbb{Z})$  with  $\mathbb{Z}[k] = \mathbb{Z}[v_1, \dots, v_k]$ . The Weyl group  $W_{SO(2k+1)}$  is the wreath product  $\mathbb{Z}/2 \wr \Sigma_k$  where  $(\mathbb{Z}/2)^k$  acts on  $T^k$  via coordinate wise complex conjugation and  $\Sigma_k$  via permutations of the coordinates. The rational cohomology of  $BSO(2k+1)$  is then given by

$$H^*(BSO(2k+1); \mathbb{Q}) \cong \mathbb{Q}[k]^{\mathbb{Z}/2 \wr \Sigma_k} \cong \mathbb{Q}[p_1, \dots, p_k].$$

The classes  $p_i$  are already defined over  $\mathbb{Z}$ . On the one hand  $p_i \in H^{4i}(BSO(2k+1); \mathbb{Z})$  is the universal  $i$ th Pontrjagin class for oriented bundles and on the other hand  $p_i = (-1)^i \sigma_i(v_1^2, \dots, v_k^2) \in \mathbb{Z}[k]^{\mathbb{Z}/2 \wr \Sigma_k}$  is up to a sign the  $i$ th elementary symmetric polynomial in the squares of the generators of  $\mathbb{Z}[k]$ . In particular, for an oriented  $(2k+1)$ -dimensional real vector bundle  $\rho$  over a space  $X$ , the total Pontrjagin class  $p(\rho) = 1 + \sum_{i=1}^k p_i(\rho)$  determines completely the map  $H^*(BSO(2k+1); \mathbb{Q}) \rightarrow H^*(X; \mathbb{Q})$  induced by the classifying map  $\rho: X \rightarrow BSO(2k+1)$ .

**Example 4.2** Let  $\rho: BT^s \rightarrow BSO(2k+1)$  be the composition of a coordinate wise inclusion  $\hat{\rho}: BT^s \rightarrow BT^k$  followed by the maximal torus inclusion  $BT^k \rightarrow BSO(2k+1)$ . Then the total Pontrjagin class of  $\rho$  is given by  $p(\rho) = \prod_{i=1}^s (1 - v_i^2)$ , where we identify  $H^*(BT^s; \mathbb{Z})$  with  $\mathbb{Z}[v_1, \dots, v_s]$ .

By Theorem 4.1, up to homotopy every vector bundle  $\omega: BT^s \rightarrow BSO(2k+1)$  is the composition of a lift  $\hat{\omega}: BT^s \rightarrow BT^k$  and  $Bj$ . If  $p(\omega) = p(\rho)$  then both maps  $\rho$  and  $\omega$  are homotopic and the underlying homomorphisms  $j_\omega, j_\rho: T^s \rightarrow T^k$  of the lifts  $\hat{\omega}$  and  $\hat{\rho}$  differ only by an element of the Weyl group (Theorem 4.1). In particular, since  $j_\rho$  is given by a coordinatewise inclusion, the homomorphism  $j_\omega$  also is a coordinate wise inclusion possibly followed by complex conjugation on some coordinates.

**Proof of Theorem 1.1** If  $K \rightarrow \Delta[r]$  is an  $r$ -coloring, Corollary 3.2 provides the appropriate splitting for  $\lambda$ . The splitting conditions on  $\lambda$  can be put into a hierarchy, a splitting of  $\lambda$  establishes a splitting of  $\lambda_{\mathbb{R}}$  and a stable isomorphism between  $\lambda$  and a direct sum of  $r$  complex line bundles, and the two latter conditions a stable isomorphism between  $\lambda_{\mathbb{R}}$  and a direct sum of  $r$  2-dimensional real bundles. It is only left to show that this last stable isomorphism allows to construct a coloring.

We will work with the model  $c(K)$  for  $DJ(K)$  and again describe vector bundles over  $c(K)$  by their classifying maps. In particular, for  $t \geq 2m$  the real vector bundle  $\rho_t := \lambda_{\mathbb{R}} \oplus \mathbb{R}^{t-2m}$  is a map  $\rho_t: c(K) \rightarrow BO(t)$ . Since we are considering stable splittings, we can pass from  $\rho_t$  to  $\rho_{t+1}$ , if necessary, and assume that  $t = 2s+1$  is odd. This will simplify the discussion. For example, if  $t$  is odd, we have  $BO(t) \simeq BSO(t) \times B\mathbb{Z}/2$ . And since  $c(K)$  is simply connected, the bundle  $\rho_t$  has a unique orientation given by the first coordinate of the map  $\rho_t: c(K) \rightarrow BSO(t) \times B\mathbb{Z}/2$ . We also denote this map by  $\rho_t$ . The total Pontrjagin class of  $\rho_t$  is given by  $p(\rho_t) = \prod_{i=1}^m (1 - v_i^2)$  [2, Sect. 6].

Let  $\phi: BT^r \rightarrow BSO(t)$  be the map induced by the composition of the coordinate wise inclusion  $T^r \subset T^s$  into the first  $r$  coordinates followed by the maximal torus inclusion  $T^s = T_{SO(t)} \subset SO(t)$ . A splitting  $\rho_t \cong (\bigoplus_{j=1}^r \theta_j) \oplus \mathbb{R}^{t-2r}$  establishes a map  $\hat{\rho}_t: c(K) \rightarrow BT^r$  such that  $\phi \hat{\rho}_t \simeq \rho_t$ .

Now let  $\alpha \in K$  be a face and  $h_\alpha: BT^\alpha \rightarrow c(K)$  the associated map. The composition  $\hat{\rho}_t h_\alpha: BT^\alpha \rightarrow BT^r$  determines a unique homomorphism  $j_\alpha: T^\alpha \rightarrow T^r$ . The total Pontrjagin class of  $\phi \hat{\rho}_t h_\alpha$  is given by  $p(\phi \hat{\rho}_t h_\alpha) = \prod_{i \in \alpha} (1 - v_i^2)$ . Example 4.2 shows that  $j_\alpha$  is a coordinate wise inclusion  $T^\alpha \rightarrow T^k$  possibly followed by complex conjugation on some coordinates. The coordinate wise inclusion defines an injection  $\alpha \rightarrow [r]$ . Since for any inclusion of faces  $\beta \subset \alpha$  the restriction  $(\hat{\rho}_t h_\alpha)|_{BT^\beta}$  equals the composition  $\hat{\rho}_t h_\beta$ , the underlying homomorphisms satisfies the formula  $j_\alpha|_\beta = j_\beta$ . We can conclude that the collection of all these maps defines a map  $[m] \rightarrow [r]$ , whose restriction to any face of  $K$  is an injection. This establishes a non degenerate simplicial map  $K \rightarrow \Delta[r]$  which is an  $r$ -coloring for  $K$ .  $\square$

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